MA110 - Engineering Mathematics-1
Problem Sheet - 11

Substitution Methods

1. Find the image of the rectangle with the given corners under the following transformations. Also evaluate the Jacobian $J(u, v)$.

   (a) $x = 2u + 3v, y = u - v$; $(0, 0), (3, 0), (3, 1), (0, 1)$. (*Ans:* Parallelogram & $J = -5$)

   (b) $x = u^2 + v^2, y = v$; $(0, 0), (1, 0), (0, 1), (1, 1)$. (*Ans:* $\{(x, y)/y^2 \leq x \leq y^2 + 1, 0 \leq y \leq 1\}$ & $J = 2u$)

   (c) $x = u, y = u^2 - v^2$; $(0, 0), (3, 0), (3, 1), (0, 1)$. (*Ans:* $\{(x, y)/x^2 - 1 \leq y \leq x^2, 0 \leq x \leq 3\}$ & $J = -2v$)

2. Let R be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 1$, $xy = 9$ and the lines $y = x$, $y = 4x$. Use the transformation $x = u/v$, $y = uv$ with $u > 0$ and $v > 0$ to rewrite $\displaystyle\iint\limits_R \left(\sqrt{\dfrac{x}{y}} + \sqrt{xy} \right) \, dx \, dy$ as an integral over an appropriate region G in the uv-plane. Then evaluate the uv-integral over G. (*Ans:* $8 + \frac{52\log 2}{3}$)

3. Evaluate $\displaystyle\iint\limits_R \ln\frac{x+y}{x-y} \, dA$, where R is the triangle with vertices $(1, 0), (4, 0), (4, 3)$. (*Ans:* 3.15669)

4. Find the volume of material cut from the solid sphere $r^2 + z^2 \leq 9$ by the cylinder $r = 3\sin\theta$. (*Ans:* 9π)

5. Find the volume of the solid D cut from the sphere $x^2 + y^2 + z^2 = 1$ by the cylinder $x^2 + (y - 1/2)^2 = 1/4$. (*Ans:* $\frac{4}{3}(\frac{\pi}{2} - \frac{2}{3})$)

6. Given $D = \{(x, y) \in \mathbb{R}^2/1 \leq 2(x + y) \leq 2, x \geq 0, y \geq 0\}$. Evaluate $\displaystyle\iint_D \frac{y}{x+y} \, dA$ using appropriate substitution. (*Ans:* $\frac{3}{16}$)

7. Given $D = \{(x, y, z) \in \mathbb{R}^3/x \geq 0, y \geq 0, z \geq 0, x \leq y + z \text{ and } 1 \leq 2(x + y + z) \leq 2\}$. Consider $f : D \to \mathbb{R}$ defined by $f(x, y, z) = \frac{z}{y+z}$. Evaluate $\displaystyle\iiint_D f(x, y, z) \, dV$ using the substitution $u = x + y + z$, $v = \frac{y+z}{x+y+z}$, $w = \frac{z}{y+z}$. (*Ans:* $\frac{7}{128}$)

8. Evaluate
$$\int \int \int |xyz| \, dx \, dy \, dz$$
   over the solid ellipsoid
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1,$$
   using a suitable substitution.

9. Find the volume of material cut from the solid sphere $r^2 + z^2 \leq 9$ by the cylinder $r = 3\sin\theta$.

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